

Madras College Maths Department  
Higher Maths  
Apps 1.2 The Circle

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Written solutions for each exercise are available at

[http://madrasmaths.com/courses/higher/revision\\_materials\\_higher.html](http://madrasmaths.com/courses/higher/revision_materials_higher.html)

You should check your solutions at the end of each exercise and ask your teacher or attend study support if there any problems.

## Representing a Circle

The equation of a circle with centre  $(a, b)$  and radius  $r$  units is:

$$(x - a)^2 + (y - b)^2 = r^2.$$

This is given in the exam.

For example, the circle with centre  $(2, -1)$  and radius 4 units has equation:

$$(x - 2)^2 + (y + 1)^2 = 4^2$$

$$(x - 2)^2 + (y + 1)^2 = 16.$$

Note that the equation of a circle with centre  $(0, 0)$  is of the form  $x^2 + y^2 = r^2$ , where  $r$  is the radius of the circle.

### EXAMPLES

1. Find the equation of the circle with centre  $(1, -3)$  and radius  $\sqrt{3}$  units.

2. A is the point  $(-3, 1)$  and B  $(5, 3)$ .

Find the equation of the circle which has AB as a diameter.

## Testing a Point

Given a circle with centre  $(a, b)$  and radius  $r$  units, we can determine whether a point  $(p, q)$  lies within, outwith or on the circumference using the following rules:

$$(p - a)^2 + (q - b)^2 < r^2 \Leftrightarrow \text{the point lies within the circle}$$

$$(p - a)^2 + (q - b)^2 = r^2 \Leftrightarrow \text{the point lies on the circumference of the circle}$$

$$(p - a)^2 + (q - b)^2 > r^2 \Leftrightarrow \text{the point lies outwith the circle.}$$

### EXAMPLE

A circle has the equation  $(x - 2)^2 + (y + 5)^2 = 29$ .

Determine whether the points  $(2, 1)$ ,  $(7, -3)$  and  $(3, -4)$  lie within, outwith or on the circumference of the circle.

## The General Equation of a Circle

The equation of any circle can be written in the form

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

where the centre is  $(-g, -f)$  and the radius is  $\sqrt{g^2 + f^2 - c}$  units.

This is given in the exam.

Note that the above equation only represents a circle if  $g^2 + f^2 - c > 0$ , since:

- if  $g^2 + f^2 - c < 0$  then we cannot obtain a real value for the radius, since we would have to square root a negative;
- if  $g^2 + f^2 - c = 0$  then the radius is zero – the equation represents a point rather than a circle.

### EXAMPLE

1. Find the radius and centre of the circle with equation

$$x^2 + y^2 + 4x - 8y + 7 = 0.$$

2. Find the radius and centre of the circle with equation

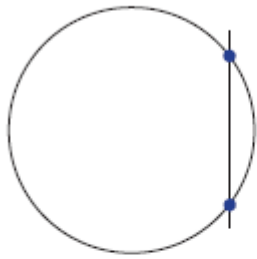
$$2x^2 + 2y^2 - 6x + 10y - 2 = 0.$$

3. Explain why  $x^2 + y^2 + 4x - 8y + 29 = 0$  is not the equation of a circle.

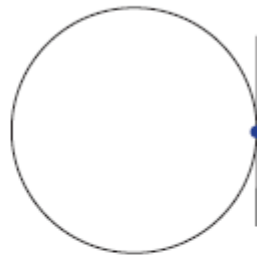
4. For which values of  $k$  does  $x^2 + y^2 - 2kx - 4y + k^2 + k - 4 = 0$  represent a circle?

## Intersection of a Line and a Circle

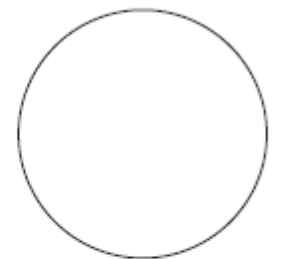
A straight line and circle can have two, one or no points of intersection:



two intersections



one intersection



no intersections

If a line and a circle only touch at one point, then the line is a **tangent** to the circle at that point.

To find out how many times a line and circle meet, we can use substitution.

### EXAMPLES

1. Find the points where the line with equation  $y = 3x$  intersects the circle with equation  $x^2 + y^2 = 20$ .

2. Find the points where the line with equation  $y = 2x + 6$  and circle with equation  $x^2 + y^2 + 2x + 2y - 8 = 0$  intersect.

## Tangents to Circles

As we have seen, a line is a tangent if it intersects the circle at only one point.

To show that a line is a tangent to a circle, the equation of the line can be substituted into the equation of the circle, and solved – there should only be one solution.

### EXAMPLE

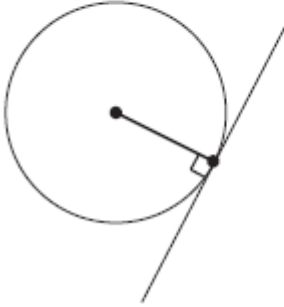
Show that the line with equation  $x + y = 4$  is a tangent to the circle with equation  $x^2 + y^2 + 6x + 2y - 22 = 0$ .



## Equations of Tangents to Circles

If the point of contact between a circle and a tangent is known, then the equation of the tangent can be calculated.

If a line is a tangent to a circle, then a radius will meet the tangent at right angles. The gradient of this radius can be calculated, since the centre and point of contact are known.



Using  $m_{\text{radius}} \times m_{\text{tangent}} = -1$ , the gradient of the tangent can be found.

The equation can then be found using  $y - b = m(x - a)$ , since the point is known, and the gradient has just been calculated.

### EXAMPLE

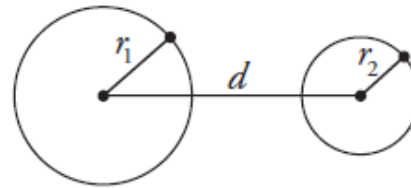
Show that  $A(1, 3)$  lies on the circle  $x^2 + y^2 + 6x + 2y - 22 = 0$  and find the equation of the tangent at A.

## Intersection of Circles

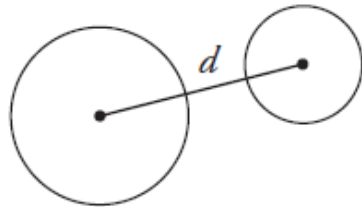
A

Consider two circles with radii  $r_1$  and  $r_2$  with  $r_1 > r_2$ .

Let  $d$  be the distance between the centres of the two circles.

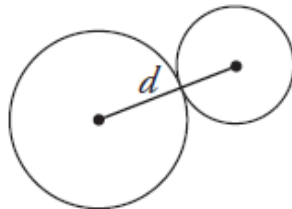


$$d > r_1 + r_2$$



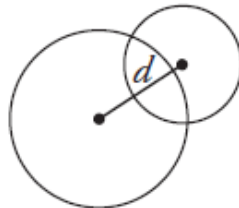
The circles do not touch.

$$d = r_1 + r_2$$



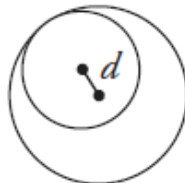
The circles touch externally.

$$r_1 - r_2 < d < r_1 + r_2$$



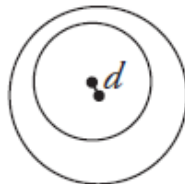
The circles meet at two distinct points.

$$d = r_1 - r_2$$



The circles touch internally.

$$d < r_1 - r_2$$



The circles do not touch.

### Note

Don't try to memorise this, just try to understand why each one is true.

**EXAMPLES**

1. Circle P has centre  $(-4, -1)$  and radius 2 units, circle Q has equation  $x^2 + y^2 - 2x + 6y + 1 = 0$ . Show that the circles P and Q do not touch.

2. Circle R has equation  $x^2 + y^2 - 2x - 4y - 4 = 0$ , and circle S has equation  $(x - 4)^2 + (y - 6)^2 = 4$ . Show that the circles R and S touch externally.

Practice Unit AssessmentsPractice 1

- 1) A circle has centre (3, 4) and radius 6 units.
- a) Write down the equation of the circle. (1)
- b) Does the point (5, 7) lie on the circle? (2)
- 2) Determine whether the line  $y = x - 1$  is a tangent to the circle
- $$x^2 + y^2 + 4x - 8y - 17 = 0 \quad (4)$$

Practice 2

- 3) A circle has centre (-2, 7) and radius 10 units.
- c) Write down the equation of the circle. (1)
- d) Does the point (4, -1) lie on the circle? (2)
- 4) Determine whether the line  $y = x + 2$  is a tangent to the circle
- $$x^2 + y^2 + x + 3y - 2 = 0 \quad (4)$$

## Practice Unit Assessment Solutions

# Test 1 Circle Practice Unit

$$\textcircled{1} (a) (x-3)^2 + (y-4)^2 = 36$$

$$(b) (5, 7) \quad x=5 \quad y=7$$

$$\begin{aligned} & (5-3)^2 + (7-4)^2 \\ &= 2^2 + 3^2 \\ &= 13 \end{aligned}$$

$13 \neq 36$ . no the point does not lie on the circle.

$$\textcircled{2} \text{ sub in } y=x-1$$

$$x^2 + (x-1)^2 + 4x - 8(x-1) - 17 = 0$$

$$x^2 + x^2 - 2x + 1 + 4x - 8x + 8 - 17 = 0$$

$$2x^2 - 10x - 8 = 0$$

$$2(x^2 - 5x - 4) = 0$$

does not factorise check  $b^2 - 4ac$ .

$$(-10)^2 - 4(2)(-8)$$

$$= 164$$

$b^2 - 4ac \neq 0 \therefore$  not a tangent.

$$\text{Test 2} \textcircled{1} (a) (x+2)^2 + (y-7)^2 = 100$$

$$(b) (4, -1) \quad x=4 \quad y=-1$$

$$\begin{aligned} & (4+2)^2 + (-1-7)^2 \\ &= 6^2 + 8^2 \\ &= 100 \end{aligned}$$

$100 = 100$ . yes point lies on circle

$$\textcircled{2} x^2 + y^2 + 2x + 3y - 2 = 0$$

$$x^2 + (x+2)^2 + 2x + 3(x+2) - 2 = 0$$

$$x^2 + x^2 + 4x + 4 + 2x + 3x + 6 - 2 = 0$$

$$2x^2 + 8x + 8 = 0$$

$$2(x^2 + 4x + 4) = 0$$

$$2(x+2)(x+2) = 0$$

$x = -2 \quad x = -2$  repeated point of contact  $\therefore$  tangent

$$\text{or } b^2 - 4ac$$

$$= 8^2 - 4(2)(8)$$

$$= 0$$

$b^2 - 4ac = 0 \therefore$  tangent